Modern approaches to quantum gravity

Solution 11 Fall 2025

1. Scalar field in AdS

(a) The separability of the wave function $\phi(t, \theta, \Omega)$ follows from the form of the Casimir $\Box$ in AdS. First, use that

$$\Box\phi = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\phi) \tag{1}$$

and write $g_{\mu\nu}$ in the form

$$g_{\mu\nu} = \begin{pmatrix} -\frac{1}{\cos^2\theta} & 0 & 0 \\ 0 & \frac{1}{\cos^2\theta} & 0 \\ 0 & 0 & \frac{\sin^2\theta}{\cos^2\theta}h_{ij} \end{pmatrix} \tag{2}$$

where h_{ij} is the metric on the unit S^{d-1} sphere. Note that $\sqrt{|g|} = \frac{\sin^{d-1}\theta}{\cos^{d+1}\theta}\sqrt{|h|}$. Thus,

$$\begin{aligned}\Box\phi = &-\cos^2\theta\partial_t^2\phi + \frac{\cos^{d+1}\theta}{\sin^{d-1}\theta}\partial_r\left(\frac{\sin^{d-1}\theta}{\cos^{d+1}\theta}\cos^2\theta\partial_r\phi\right) \\ &+\frac{\cos^2\theta}{\sin^2\theta}\underbrace{\frac{1}{\sqrt{|h|}}\partial_i(\sqrt{h}h^{ij}\partial_j\phi)}_{=\Delta_{S^{d-1}}\phi}\end{aligned} \tag{3}$$

where we recognized the sphere Laplacian $\Delta_{S^{d-1}}\phi$. This form is precisely of the form

$$\Box = \Box_t + \Box_r + \Box_{S^{d-1}} \tag{4}$$

The most general solution is given by a linear combination of solutions of the form $A(t)B(\theta)C(\Omega)$. The nice way to obtain solutions is to diagonalize each operator. We diagonalize ∂_t^2 in the standard way,

$$\partial_t^2\phi = -\omega^2\phi \tag{5}$$

where ω is the energy. The eigenfunctions of $\Delta_{S^{d-1}}$ are represented in spherical harmonics $Y_\ell(\Omega)$ (there are other quantum numbers as well. For example, in the case $d-1=2$, there would be $m = -\ell, ..., \ell$) which obey the property

$$\Delta_{S^{d-1}}Y_\ell(\Omega) = -\ell(\ell+d-2)Y_\ell(\Omega) \tag{6}$$

You may be familiar with the case $d-1=2$, where $\Delta_{S^2} = -J^2$, $J^2 = \ell(\ell+1)$

(b) After having diagonalized $\partial_t^2 \to -\omega^2$ and $\Delta_{S^{d-1}} \to -\ell(\ell+d-2)$, $\Box\phi = m^2\phi$ reduces to

$$\cos^2\theta\, G''(\theta) + \frac{d-1}{\tan\theta}G'(\theta) + \left(\omega^2\cos^2\theta - \frac{\ell(\ell+d-2)}{\tan^2\theta}\right)G(\theta) = m^2 G(\theta) \quad (7)$$

In the mathematica file provided, we find two independent solutions to this equation, namely

$$G_1(\theta) = \cos^{d-\Delta}\theta\ \sin^\ell\theta\ {}_2F_1\left(\frac{d+\ell-\Delta+\omega}{2}, \frac{d+\ell-\Delta-\omega}{2}, \frac{d}{2}-\Delta+1, \cos^2\theta\right) \quad (8)$$

$$G_2(\theta) = \cos^{\Delta}\theta\ \sin^\ell\theta\ {}_2F_1\left(\frac{\ell+\Delta+\omega}{2}, \frac{\ell+\Delta-\omega}{2}, -\frac{d}{2}+\Delta+1, \cos^2\theta\right) \quad (9)$$

These are not the two linearly independent solutions we were looking for, since we were looking for $\sin^2\theta$ as the last argument of ${}_2F_1$. To see what happens, we need to use hypergeometric function identities (c.f. wikipedia), including

$$\begin{aligned}{}_2F_1(a,b;c;z) = &\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}{}_2F_1(a,b;a+b+1-c;1-z) \\ &+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}{}_2F_1(c-a,c-b;1+c-a-b;1-z).\end{aligned} \quad (10)$$

Choosing $a = \frac{\ell+\Delta+\omega}{2}$, $b = \frac{\ell+\Delta-\omega}{2}$ and $c = -\frac{d}{2}+\Delta+1$, $z = \cos^2\theta$, $1-z = \sin^2\theta$, this implies that the geometric ${}_2F_1$ appearing in G_2 can be expressed as

$$\begin{aligned}{}_2F_1(a,b,c,\cos^2\theta) = A\ {}_2F_1\left(\frac{\ell+\Delta+\omega}{2}, \frac{\ell+\Delta-\omega}{2}, \ell+\frac{d}{2}, \sin^2\theta\right) \\ + B\sin^{-d-2l+2}\theta\ {}_2F_1\left(\frac{-d+\Delta-\ell+\omega+2}{2}, \frac{-d+\Delta-\ell-\omega+2}{2}, -\frac{d}{2}-\ell+2, \sin^2\theta\right)\end{aligned} \quad (11)$$

where A and B are come constants, independent of θ. Plugging this in G_2 this implies that

$$G_2(\theta) = AG_+(\theta) + BG_-(\theta) \quad (12)$$

where

$$G_+ = (\sin\theta)^\ell(\cos\theta)^\Delta{}_2F_1(\frac{\Delta+\ell+\omega}{2}, \frac{\Delta+\ell-\omega}{2}; \ell+\frac{d}{2}; \sin^2\theta) \quad (13)$$

$$G_- = (\sin\theta)^{(2-d-\ell)}(\cos\theta)^\Delta{}_2F_1(\frac{\Delta+2-d-\ell+\omega}{2}, \frac{\Delta+2-d-\ell-\omega}{2}; 2-\ell-\frac{d}{2}; \sin^2\theta) \quad (14)$$

The same analysis can be carried to G_1 with the use of the identity ${}_2F_1(a,b,c,z) = (1-z)^{c-a-b}{}_2F_1(c-a, c-b, c, z)$, as it should be, since G_+ and G_- are already independent solutions.

At $\theta \to 0$, ${}_2F_1(a,b,c,\sin^2\theta) = 1$ and $\frac{d}{d\theta}\ {}_2F_1(a,b,c,\sin^2\theta) = 0$ (where a, b, c are arbitrary), and similarily for $\cos^\Delta$. So the divergent behaviour of G_- only comes from $(\sin\theta)^{2-d-l}$,

$$\frac{d}{d\theta}G_-|_{\theta\to 0} \sim (2-d-\ell)\sin(\theta)^{(1-d-\ell)} \quad (15)$$

Since $d > 1$ (the case of AdS_2 is special and we won't consider it), this diverges generically as $\theta \to 0$.

(c) The unit vector n^i pointing outwards the sphere has a non-zero component $n^\theta > 0$. To fix it, we impose the normalization

$$n^\mu g_{\mu\nu} n^\nu = 1 \implies n^\theta = \cos\theta \tag{16}$$

The induced metric on the fixed θ sphere boundary which we will denote by $\mathcal{B}$ is obtained by setting $d\theta = dt = 0$ in ds^2, thus

$$ds^2|_{\mathcal{B}} = \tan^2\theta \; h_{ij} d\alpha^i d\alpha^j \tag{17}$$

where α^i are $d-1$ angles on the unit sphere and h_{ij} is the metric on the unit sphere. This implies that

$$\sqrt{|g|_{\mathcal{B}}} = \sqrt{|h|} \tan^{d-1}\theta \tag{18}$$

Combining both, this gives

$$\int_{S^{d-1}} d^{d-1}\alpha \sqrt{|g|} n^i T_{it} = \int_{S^{d-1}} d^{d-1}\alpha \sqrt{|h|} \tan^{d-1}\theta \cos\theta \; T_{\theta t} \tag{19}$$

Note that $\tan^{d-1}\theta\cos\theta = \tan^{d-2}\theta \; \sin\theta$ and $\sin\theta \to 1$ as $\theta \to \pi/2$. Thus, the condition that it vanishes at the boundary is simply

$$\tan^{d-2}\theta \; T_{\theta t} \to 0 \qquad (\theta \to \pi/2) \tag{20}$$

Noting that $g_{\theta t} = 0$, $R_{\theta t} = 0$ (this follows from the fact that AdS_{d+1} is maximally symmetric, and thus $R_{\mu\nu} \propto g_{\mu\nu}$), the (θt) component of the stress tensor reads

$$T_{\theta t} = 2\partial_\theta\phi\partial_t\phi - \beta\nabla_\theta\nabla_t\phi^2 \tag{21}$$

Using $\partial_t\phi = -\omega^2\phi$ and $\Gamma^t_{\theta t} = \tan\theta$ is the only non-vanishing Christoffel with (θt) lower indices we obtain

$$T_{\theta t} = -\omega^2\partial_\theta\phi^2 + 2\beta\omega^2\partial_\theta\phi^2 - 2\beta\tan\theta\omega^2\phi^2 \tag{22}$$

Thus, the condition can be written as

$$(\tan\theta)^{d-2}[(1-2\beta)\partial_\theta + 2\beta\tan\theta]G_+^2 \to 0 \qquad (\theta \to \pi/2) \tag{23}$$

We now investigate the behaviour of G_+ when $\sin\theta \to 1$, $\cos\theta \to 0$. Let us define the arguments of the hypergeometric ${}_2F_1$ as

$$a \equiv \frac{\Delta + \ell + \omega}{2} \qquad b \equiv \frac{\Delta + \ell - \omega}{2} \qquad c \equiv \ell + \frac{d}{2} \tag{24}$$

One property of the hypergeometric is that, when $c < a + b$ (which is true in our case), the limit $z \to 1$ reads

$${}_2F_1(a,b,c,z) \sim (1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \tag{25}$$

Thus as $\sin\theta \to 1$,

$$\begin{aligned} G_+^2 &\sim (\sin\theta)^{2\ell}(\cos\theta)^{2\Delta}(\cos^2\theta)^{2(c-a-b)}\frac{\Gamma(c)^2\Gamma(a+b-c)^2}{\Gamma(a)^2\Gamma(b)^2} \\ &\sim (\cos\theta)^{2(d-\Delta)}\frac{\Gamma(c)^2\Gamma(a+b-c)^2}{\Gamma(a)^2\Gamma(b)^2} \end{aligned} \tag{26}$$

(we dropped $\sin\theta \sim 1$). For $\partial_\theta G_+^2$, one of the derivatives can hit $\cos\theta$ generating the term

$$\partial_\theta G_+^2 \sim (\cos\theta)^{2(d-\Delta)-1}\frac{\Gamma(c)^2\Gamma(a+b-c)^2}{\Gamma(a)^2\Gamma(b)^2} \tag{27}$$

Thus, the condition (22) (using $(\tan\theta) \sim (\cos\theta)^{-1}$) reduces to

$$(\cos\theta)^{d-2\Delta+1}\frac{\Gamma(c)^2\Gamma(a+b-c)^2}{\Gamma(a)^2\Gamma(b)^2} \to 0 \tag{28}$$

Note that $d-2\Delta+1 \leq -d+1$, so $(\cos\theta)^{d-2\Delta+1} \to \infty$. We thus need

$$\frac{\Gamma(c)^2\Gamma(a+b-c)^2}{\Gamma(a)^2\Gamma(b)^2} = 0 \tag{29}$$

This is true whenever a or b is a pole of the gamma function (the gamma function has no zeros), namely

$$a = -n \quad \text{or} \quad b = -n \qquad n = 0, 1, ... \tag{30}$$

This implies

$$\Delta + \ell \pm \omega = -2n \tag{31}$$

meaning

$$\pm\omega = \Delta + \ell + 2n \tag{32}$$

(d) For ω to be real, we require Δ to be real. This means

$$d^2 + 4m^2 > 0 \implies m^2 \geq -\frac{d^2}{4}\,. \tag{33}$$

Plugging back the AdS radius R, this would give

$$m^2R^2 \geq -\frac{d^2}{4}\,. \tag{34}$$

This is known as the Breitenlohner-Freedman bound.

2. Lowest scalar energy state in AdS

(a) *(optional part)* The CFT generators are expressed in terms of the $SO(d,2)$ generators as

$$D = -J_{0,d+1}, \quad M_{\mu\nu} = J_{\mu\nu}, \tag{35}$$
$$P_\mu = J_{\mu 0} + iJ_{\mu,d+1}, \quad K_\mu = J_{\mu 0} - iJ_{\mu,d+1}, \tag{36}$$

with

$$J_{MN} = -i\left(X_M\frac{\partial}{\partial X^N} - X_N\frac{\partial}{\partial X^M}\right). \tag{37}$$

where X^A are the embedding coordinates which in global coordinates read

$$X^0 = R\cos t \cosh\rho, \tag{38}$$
$$X^\mu = R\Omega^\mu \sinh\rho, \tag{39}$$
$$X^{d+1} = -R\sin t\cosh\rho. \tag{40}$$

It's easiest to express the derivative $\partial/\partial t$ in terms of the $\partial/\partial X^A$ (and likewise for $\partial/\partial\rho$ and ∇_μ) instead of working the other way around. For instance, we have

$$\frac{\partial}{\partial t} = \frac{\partial X^A}{\partial t}\frac{\partial}{\partial X^A} = -R\sinh\rho\frac{\partial}{\partial X^0} - R\cosh t\cosh\rho\frac{\partial}{\partial X^{d+1}}. \tag{41}$$

But this is precisely

$$\frac{\partial}{\partial t} = X^{d+1}\frac{\partial}{\partial X^0} - X^0\frac{\partial}{\partial X^{d+1}} = -X_{d+1}\frac{\partial}{\partial X^0} + X_0\frac{\partial}{\partial X^{d+1}} = iJ_{0,d+1}, \tag{42}$$

since $X_0 = -X^0$ and $X_{d+1} = -X^{d+1}$. Hence $-J_{0,d+1} = i\partial_t$, as claimed.

Next, if we extend the Ω^μ to cover all of $\mathbb{R}^d$, we have

$$\frac{\partial}{\partial\Omega^\mu} = \frac{X^A}{\partial\Omega^\mu}\frac{\partial}{\partial X^A} = R\sinh\rho\frac{\partial}{\partial X^\mu}, \quad \mu = 1,\ldots,d, \tag{43}$$

so more properly speaking, we have the $\mathbb{S}^{d-1}$-covariant derivatives

$$\nabla_\mu = R\sinh\rho\left[\frac{\partial}{\partial X^\mu} - \Omega_\mu\Omega^\nu\frac{\partial}{\partial X^\nu}\right] \tag{44}$$

which indeed obey $\Omega^\mu\nabla_\mu = 0$. Consequently

$$X_\mu\frac{\partial}{\partial X_\nu} = R\sinh\rho\,\Omega_\mu\frac{\partial}{\partial X^\nu} = \Omega_\mu\nabla_\nu + \Omega_\mu\Omega_\nu\Omega^\rho\frac{\partial}{\partial X^\rho}, \tag{45}$$

so (since the second term is symmetric in $\mu \leftrightarrow \nu$)

$$J_{\mu\nu} = -i\left(\Omega_\mu\nabla_\nu - \Omega_\nu\nabla_\mu\right) = M_{\mu\nu}. \tag{46}$$

Finally, we have

$$\frac{\partial}{\partial\rho} = \frac{\partial X^A}{\partial\rho}\frac{\partial}{\partial X^A} = R\left[\cos t\sinh\rho\frac{\partial}{\partial X^0} + \Omega^\mu\cosh\rho\frac{\partial}{\partial X^\mu} - \sin t\sinh\rho\frac{\partial}{\partial X^{d+1}}\right]. \tag{47}$$

We are aiming for an expression of the form J_{AB}, expressed purely in terms of the X_A. As a first step, we can introduce the operators

$$\mathcal{D}_\mu = \Omega_\mu\frac{\partial}{\partial\rho} + \frac{1}{\tanh\rho}\nabla_\mu \tag{48}$$

which are somewhat simpler: they can be recast as

$$\mathcal{D}_\mu = R\left[\cos t\sinh\rho\,\Omega_\mu\frac{\partial}{\partial X^0} + \cosh\rho\frac{\partial}{\partial X^\mu} - \Omega_\mu\sin t\sinh\rho\frac{\partial}{\partial X^{d+1}}\right], \tag{49}$$

$$= X_\mu \left(\cos t \frac{\partial}{\partial X^0} - \sin t \frac{\partial}{\partial X^{d+1}} \right) + R \cosh \rho \frac{\partial}{\partial X^\mu}. \tag{50}$$

Moreover

$$\Omega^\mu \tanh \rho \frac{\partial}{\partial t} = -X_\mu \left(\sin t \frac{\partial}{\partial X^0} + \cos t \frac{\partial}{\partial X^{d+1}} \right). \tag{51}$$

Using the identity $R \cosh \rho = -e^{\pm it}(X_0 \pm iX_{d+1})$, we can therefore write

$$\mathcal{D}_\mu - i\Omega_\mu \tanh \rho \frac{\partial}{\partial t} = e^{it} \left[X_\mu \left(\frac{\partial}{\partial X^0} + i \frac{\partial}{\partial X^{d+1}} \right) - (X_0 + iX_{d+1}) \frac{\partial}{\partial X^\mu} \right], \tag{52}$$

that is to say

$$-ie^{-it} \left(\mathcal{D}_\mu - i\Omega_\mu \tanh \rho \frac{\partial}{\partial t} \right) = J_{\mu 0} + iJ_{\mu,d+1}. \tag{53}$$

Likewise

$$\mathcal{D}_\mu + i\Omega_\mu \tanh \rho \frac{\partial}{\partial t} = e^{-it} \left[X_\mu \left(\frac{\partial}{\partial X^0} - i \frac{\partial}{\partial X^{d+1}} \right) - (X_0 - iX_{d+1}) \frac{\partial}{\partial X^\mu} \right], \tag{54}$$

that is to say

$$-ie^{it} \left(\mathcal{D}_\mu + i\Omega_\mu \tanh \rho \frac{\partial}{\partial t} \right) = J_{\mu 0} - iJ_{\mu,d+1}. \tag{55}$$

This agrees precisely with the formulas from the exercise.

(b) The requirement $D\phi = \Delta\phi$ gives

$$i \frac{\partial}{\partial t} \phi = \Delta \phi \implies \phi(r, t, \Omega) = e^{-i\Delta t} \phi_0(r, \Omega) \tag{56}$$

The requirement $K_\mu \phi = 0$ implies that $\nabla_\mu \phi = 0$ (independent of the S^{d-1} angular coordinates), and

$$(-\partial_\rho - i \tanh \rho \partial_t)\phi_0 = 0 \implies \partial_\rho \phi_0 = -\Delta \tanh \rho \; \phi_0 \tag{57}$$

This is solved by

$$\phi_0 \propto e^{-\Delta \log \cosh \rho} \implies \phi \propto \left(\frac{e^{-it}}{\cosh \rho} \right)^\Delta \tag{58}$$

(c) The change of variables between the two coordinates system is $\tanh \rho = \sin \theta \implies \frac{1}{\cosh \rho} = \cos \theta$. So we obtained that a scalar operator ϕ which is a **primary state of the CFT algebra with dimension** Δ is expressed as

$$\phi_{\text{CFT,primary}} \propto e^{-i\Delta t} (\cos \theta)^\Delta \tag{59}$$

whereas in the last exercise we solved the **massive scalar fields in** AdS and obtained that $\phi = e^{-i\omega t} G_+(\theta) Y_\ell(\Omega)$. Choosing $\ell = 0$, $\omega = \Delta_m$ (**lowest energy state**) and $Y_{\ell=0} \propto 1$ the solutions reads

$$\phi_{AdS,\text{lowest energy state}} \propto e^{-i\Delta_m t} (\cos \theta)^{\Delta_m} {}_2F_1(1, 0, \frac{d}{2}, \sin^2 \theta) = e^{-i\Delta_m t} (\cos \theta)^{\Delta_m} \tag{60}$$

where Δ_m in this exercise was defined as $\Delta_m \equiv \frac{d}{2}(1 + \sqrt{1 + 4m^2})$. We see that the two are the same, upon identifying the CFT dimension Δ with the quantity Δ_m.

3. Vector fields in AdS

(a) Massive vector fields have $D-1$ degrees of freedom, given that the conjugate momentum to the component A^0 is $F_{00}=0$, hence A^0 is non-dynamical. Massless vectors, have $D-2$ because another one of their components can be removed with a gauge transformation.

(b) Trivial use of Euler-Lagrange's equations.

(c) For the $D-1$ components with $i \neq z$, the equations of motion yield

$$\partial_i F^{ib} + \partial_z F^{zb} + (4-D) z^{-1} F^{zb} = z^{-2} m^2 A^b, \tag{61}$$

which gives

$$\nu(\nu-1) z^{\nu-2} J_i + (3-d)\nu z^{\nu-2} J_i = m^2 z^{\nu-2} J_i. \tag{62}$$

Thus, we have the result

$$\nu(\nu-1) + (3-d)\nu = m^2. \tag{63}$$

Now we undo the conformal transformation:

$$A_i = \eta_{ij} A^j = z^2 g_{ij} A^j, \tag{64}$$

and conclude that A_i has weight 1 under dilation $x^i \mapsto \Omega x^i, z \mapsto \Omega z$, just as we would expect from a free vector field. Hence, the weight of J is

$$\Delta_J = z^{-\nu} A = 1 + \nu, \tag{65}$$

and

$$(\Delta-1)(\Delta-2) + (3-d)(\Delta-1) = m^2, \tag{66}$$

or

$$(\Delta-1)(\Delta+1-d) = m^2. \tag{67}$$

(d) For $m^2 = 0$, we find

$$\Delta = d-1 \quad \text{or} \quad \Delta = 1. \tag{68}$$

Here, $\Delta = d-1$ corresponds to a conserved boundary current J^i, and $\Delta = 1$ corresponds to a boundary potential (background source) A_i. If $A_i = 0$, then it corresponds to a globally conserved current.

(e) For A_z: For $m^2 = 0$, we have gauge symmetry $\delta A_a = \nabla_a \alpha$, so we can impose the gauge condition, e.g., $A_z = 0$, giving $D-2 = d-1$ degrees of freedom. Using the z-equation of motion:

$$\partial_i F^{iz} = 0, \quad F^{zz} = 0, \tag{69}$$

we get that J^i is a conserved current on the boundary

$$\partial_z(\partial_i A^i) = 0 \implies \partial_i J^i = \nabla_i J^i = 0. \tag{70}$$

Such conserved currents are vectors in $D-1$ dimensions, satisfying one constraint. Hence they have $D-2$ degrees of freedom, which matches bulk gauge fields.